How complicated is the one-dimensional chaos:

descriptive theory of chaos

O.M.Sharkovsky

Institute of Mathematics
National Academy of Sciences of Ukraine

June 2017

Most recently, the book of Sylvie Ruette *Chaos on the interval* (Amer. Math. Soc., ser. University lecture **67**, 2017) appeared, the aim of which is "to survey the relations between the various kinds of chaos and related notions for continuous interval maps".

However, the book does not even mention the research of this talk author published back in the sixties of the past century. This research showed a huge variety of the trajectory attractors (i.e., ω-limit sets) and proved the great complexity of their attraction basins, in particular, a very intricate interweaving of different basins.

All this is obtained with the use of descriptive set theory and gives a good idea of the complexity of the one-dimensional chaos.

[1] *On attracting and attracted sets*, Soviet Math. Dokl. **6**, 1965, 268-270 (transl. from Dokl. Akad. Nauk SSSR **160**, 1965, 1036-1038).

[2] *A classification of fixed points*, Amer. Math. Soc. Transl. (2) **97**, 1970, 159-179 (transl. from Ukrain. Mat. Zh. **17**(5), 1965, 80-95).

[3] *Behavior of a mapping in the neighborhood of an atrracting set*, Amer. Math. Soc. Transl. (2) **97**, 1970, 227-258 (transl. from Ukrain. Mat. Zh. **18**(2), 1966, 60-83).

[4] *Partially ordered system of attracting sets*, Soviet Math. Dokl. **7**, 1966, 1384-1386 (transl. from Dokl. Akad. Nauk SSSR **170**, 1966, 1276-1278).

[5] *Attractors of trajectories and their basins*, Naukova Dumka, Kiev, 2013, 320p. (in Russian).

H e r e :

attracting set = attractor of a trajectory = ω-limit set of a trajectory

attracted set = basin of an attractor = subset of the phase space consisting of all trajectories with the same ω-limit set

We consider maps $f \in C^0(I, I)$ where I is a closed interval under the condition that the topological entropy $h(f)$ is positive. This means that the one-dimensional dynamical systems given by f are complex.

As well known, the condition $h(f) > 0$ is equivalent to each from the following:

(a) f has a cycle of period $\neq 2^i, i \geq 0$,

(b) f has a homoclinic trajectory,

(c) there exist $m \geq 1$ and closed intervals $J, K \subset I$ such that $f^m J \cap f^m K \subset J \cup K$.

In the theory of dynamical systems, along with open sets (for example, basins of sinks, wandering sets) and closed sets (ω-limit sets, nonwandering sets, centers of dynamical systems), sets with more complicated structure are considered.

There appear F_σ sets, which are unions of no more than countably many closed sets, such as the set of all periodic points, G_δ sets, which are intersections of no more than countably many open sets, such as the set of all transitive points of transitive systems, $F_{\sigma\delta}$ sets, which are intersections of no more than countably many F_σ sets, etc.

We also use the Baire classification of sets according to which open sets and closed sets together with all sets being both F_σ and G_δ constitute the first class. The second class consists of sets that are either F_σ or G_δ but not both, and sets that are at the same time $F_{\sigma\delta}$ and $G_{\delta\sigma}$ but do not belong to the first class. The third class consists of sets being either $F_{\sigma\delta}$ or $G_{\delta\sigma}$ but not both, and sets that ... Further classes are defined in a similar way.

Usually upper descriptive estimates are obtained relatively easy even for dynamical systems on an arbitrary space X with countable basis of its topology, and in [1], such upper estimates for systems on arbitrary compacts was obtained. Namely:

(a) if an attractor $\mathcal{A}$ is maximal, i.e. there is no attractors $\tilde{A} \supset \mathcal{A}$, then the basin $\mathcal{B}(\mathcal{A})$ is a G_δ set;

(b) if an attractor $\mathcal{A}$ is locally maximal, i.e. there exists a neighborhood of $\mathcal{A}$, not containing attractors $\tilde{A} \supset \mathcal{A}$, then the basin $\mathcal{B}(\mathcal{A})$ is both a $F_{\sigma\delta}$ set and a $G_{\delta\sigma}$ set;

(c) in any case, basin $\mathcal{B}(\mathcal{A})$ is (no more complex than) a $F_{\sigma\delta}$ set in X, i.e. it always can be represented as an intersection of no more than countably many unions of no more than countably many closed sets.

But the proof of the accessibility of these estimates at least for a certain class of systems, and thus, the proof of the complex interlacing of the basins of investigated attractors, is really a very complicated problem even in dimension one ...

Nevertheless, as it turned out, all these estimates are accessible for one-dimensional systems when f has a cycle of period $\neq 2^i$. Namely, it was shown in [2-4] that in this case there exists a maximal attractor A_{max} containing a cycle and so containing continuum many attractors of kind (c); the basin of every such (of kind (c)) attractor is a third class set, i.e., is a $F_{\sigma\delta}$ set but not a $G_{\delta\sigma}$ set. This means that

1) here we have the very complex curved interlaced trajectories with different asymptotic behavior, and

2) from the viewpoint of descriptive theory, one-dimensional chaos is as complex as is many-dimensional or even infinity-dimensional chaos.

It can be noted that statement (a) is proved very simply, and below we offer this proof, as an example of obtaining descriptive estimates for sets.

Let Σ be a countable basis in X formed by open sets and let $\Sigma_{\mathcal{A}}$ be the part of Σ "intersecting" $\mathcal{A}$:

$$\Sigma_{\mathcal{A}} = \{\sigma_i \in \Sigma : \ \sigma_i \cap \mathcal{A} \neq \emptyset, \ \ \bigcup_{i=1}^{\infty} \sigma_i \supset \mathcal{A}\}.$$

If $x \in \mathfrak{B}(\mathcal{A})$, then $\{f^k(x)\}_{k=0}^{k=\infty}$ intersects with each σ_i, $i \geq 1$, and hence, $\mathfrak{B}(\mathcal{A}) \ \subseteq \ Q_i = \bigcup_{k=1}^{\infty} f^{-k}(\sigma_i)$, $i = 1, 2, \ldots$.

If $x \notin \mathfrak{B}(\mathcal{A})$, i.e. $\mathcal{A}_x \not\supseteq \mathcal{A}$, then there exists a point $z \in \mathcal{A}$, not belonging to $\mathcal{A}_x$, and hence there is $\sigma_{i'} \ni z$ not containing points of the trajectory $\{f^k(x)\}_{k=0}^{k=\infty}$; thus $x \notin Q_{i'}$, and $x \notin \bigcap Q_i$.

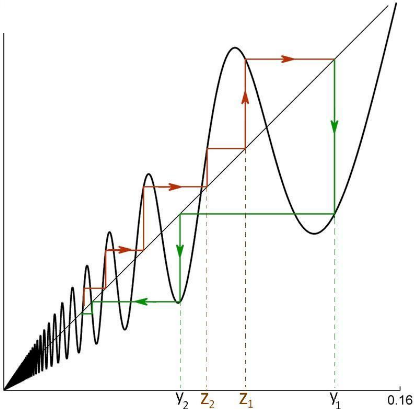
Hence, $\mathfrak{B}(\mathcal{A}) = \bigcap_{i=1}^{\infty} Q_i$ and, as far as Q_i are open sets, $\mathfrak{B}(\mathcal{A})$ is a G_δ set.

Let us dwell on statement (c).

In [2], it was proved that even basins of the simplest attractors – cycles – can be very complex, namely, can be a set of the third class in the Baire classification. Later in [3], it was shown that such situation is typical, namely, even for quadratic maps, the basin of any attractor that contains a cycle and is not maximal or locally maximal (which is typical where $h(f) > 0$), is a set of the third class.

However, the above case of cycles is somewhat exceptional, it can happen with a cycle that is limiting for cycles of the same or doubled period, for example, with maps of the same type as the map $f : x \mapsto x - x\sin(1/x))$, whose fixed point $x = 0$ is limiting for its other fixed points.

We refer to a fixed point α as a fixed point of mixed type or an attracting-repulsing fixed point if there exist two sequences of points $y_1 > y_2 > y_3 > \ldots$ and $z_1 > z_2 > z_3 > \ldots$, $i = 1, 2, \ldots$, tending to α such that $\alpha \le f(y_i) \le y_{i+1}$, $z_i \le f(z_{i+1})$.



Recall the known Baire's example of a third class set.

Let E consist of the irrational points of the interval $(0,1)$, which is a G_δ set (in the usual topology). For every point of E, there corresponds a unique continued fraction of the form

$$\frac{1}{n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \dots}}}.$$

The Baire set $\mathbb{B}$ of third class consists of points from E for which $n_j \to \infty$.

Let us take the map $g : E \to (0,1)$ defined by $g : x \mapsto \{1/x\}$, where $\{\cdot\}$ is for the fractional part of a number. The map g is continuous (on E) and $g(E) = E$. Indeed, if $x = \frac{1}{n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \dots}}}$, then $g(x) = \frac{1}{n_2 + \frac{1}{n_3 + \dots}}$.

Thus the Baire set $\mathbb{B}$ is constituted by the points $x \in E$ for which $g^j x \to 0$ as $j \to \infty$, i.e., in our notations, $\mathbb{B}$ is just the set $\mathfrak{B}(\{0\})$ — the basin of the fixed point $x = 0$.

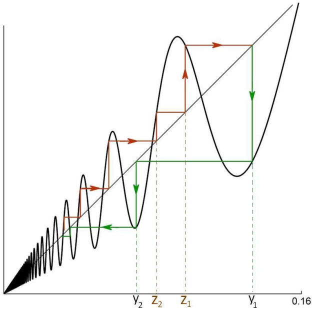
The Baire criterion for belonging a set to the third class.

Let $p_{j_1 j_2 \ldots j_k}$ be perfect nowhere dense on R or J sets, and

1) $p_{j_1 \ldots j_{k-1} j_k} \subset p_{j_1 \ldots j_{k-1}}$,
2) $p_{j_1 \ldots j_{k-1} j_k}$ is nowhere dense on $p_{j_1 \ldots, j_{k-1}}$,
3) $\bigcup\limits_{j_k=1}^{\infty} p_{j_1 \ldots j_{k-1} j_k}$ is everywhere dense on $p_{j_1 \ldots j_{k-1}}$.

The set $P = \bigcap\limits_{k=1}^{\infty} \bigcup\limits_{j_1, \ldots, j_k=1}^{\infty} p_{j_1 \ldots j_k}$ is of Baire's third class.

Theorem. *If a map has an attracting-repulsing fixed point, then the basin of this point is a third Baire class set.*



Road to chaos through the “creeping” feedback (nonsmooth realization)

We can see in this figure how the repulsion from the fixed point ($x = 0$) and the attraction to it occur (“creeping” feedback). It remains to show that the set of points x for which $f^i(x) \to 0$ when $i \to \infty$ can be represented as an union of two sets, namely, a set that satisfies to the Baire criterion for being in the third class and a set of a Baire class ≤ 2. It is this complicated problem that is solved in [2].

Theorem. *If an attractor $\mathcal{A}$ is not maximal, contains a cycle and any neighborhood of $\mathcal{A}$ contains an attractor $\tilde{\mathcal{A}} \supset \mathcal{A}$, then the basin $\mathcal{B}(\mathcal{A})$ is a third Baire's class set.*

For one-dimensional systems, **only the irreversibility of f gives** an opportunity for the **feedback, that opens the way to chaos.**

In our case, there exists a maximal attractor $\mathcal{A}_{max} \supset \mathcal{A}$ which contains points x such that $f^{-1}(x)$ consists of at least two points and in such way there is a "fast" feedback on $\mathcal{A}_{max}$. Here we already deal with a feedback not only for cycles (i.e. attractors consisting of a single trajectory), but also for more complicated attractors. The simplest variant of this is a homoclinic trajectory together with its limit "attracting-repulsing on $\mathcal{A}_{max}$" cycle or a closed heteroclinic contour including several cycles. $\mathcal{A}_{max}$ always contains such kind of attractors.

The "fast" feedback on $\mathcal{A}_{max}$ generates a "creeping" feedback for attractors which are not maximal or locally maximal.

In [3], the above theorem is proved using [2], especially the corresponding theorem for cycles.

Thus, the basin of every attractor $\mathcal{A} \subseteq \mathcal{A}_{max}$ is dense on $\mathcal{A}_{max}$, in particular, $\mathcal{B}(\mathcal{A}_{max})$ is a G_δ set of the second Baire class, for every locally maximal attractor $\mathcal{A}_{lmax}$, the basin $\mathcal{B}(\mathcal{A}_{lmax})$ is a both $F_{\sigma\delta}$ and $G_{\delta\sigma}$ set, and hence, of the second Baire class.

For any attractor $\mathcal{A} \subset \mathcal{A}_{max}$ that is not locally maximal and contains a cycle, the basin $\mathcal{B}(\mathcal{A})$ is a $F_{\sigma\delta}$ set of the third Baire class. However, basins of any two such attractors $\mathcal{A}', \mathcal{A}''$, $\mathcal{A}' \cap \mathcal{A}'' = \emptyset$, are separated by sets of the second Baire class, namely, there exist two F_σ sets $\mathcal{F}', \mathcal{F}'' \subset \mathcal{A}_{max}$, $\mathcal{F}' \cap \mathcal{F}'' = \emptyset$, such that $\mathcal{B}(\mathcal{A}') \subset \mathcal{F}'$ and $\mathcal{B}(\mathcal{A}'') \subset \mathcal{F}''$.

Of course, information about attractors themselves and their interrelationships should be an essential part of the descriptive theory of chaos.

In [4] and [5, sect.4], the aggregates $\mathbb{M}$ and $\mathbb{M}'$ of all attractors and, correspondingly, all locally maximal attractors (which are contained in $\mathcal{A}_{max}$) were considered.

$\mathbb{M}$ contains continuum many locally maximal attractors other than cycles; each of them is a Cantor set on which the periodic points set is everywhere dense.

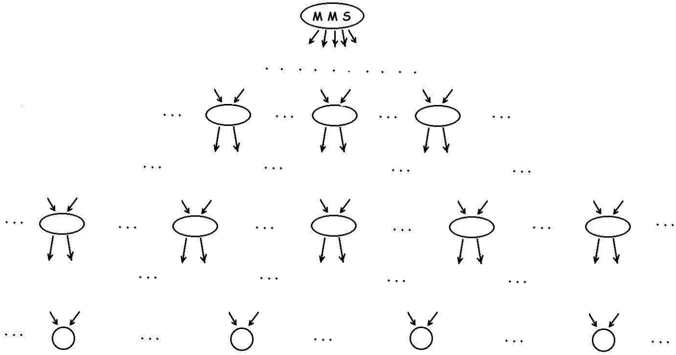
$\mathbb{M}$ contains continuum many minimal attractors other than cycles and hence being Cantor sets.

There is a natural partial ordering in $\mathbb{M}$:
if $\mathcal{A}' \subset \mathcal{A}$, then $\mathcal{A}'$ precedes $\mathcal{A}$ in $\mathbb{M}$.

The maximal attractor $\mathcal{A}_{max}$ has no immediate predecessor in any maximal chain. Every maximal chain from $\mathbb{M}$ is countable or has the power of continuum. The set of attractors immediately succeeding every attractor $\neq \mathcal{A}_{max}$ has the power of continuum.

Any maximal chain from $\mathbb{M}'$ is countable and similar to the rational numbers set :

for each $\mathcal{A}' \subset \mathcal{A}''$, there is $\mathcal{A}'''$ such that $\mathcal{A}' \subset \mathcal{A}''' \subset \mathcal{A}''$ ...



⬭ *stands for a locally maximal omega-limit set which is a Cantor set*

◯ *stands for a locally maximal omega-limit set which is a finite set (a cycle)*

each arrow ↓ *replaces the sign* $>$

Several remarks more.

In [3, theorem 2], and in [5, theorem 3.2.4], there are stated:
for any $\mathcal{A} \in \mathbb{M}$ containing a cycle, and $\varepsilon > 0$, there exists a cycle $P \in \mathbb{M}$ such that the Hausdorf distance beetween $\mathcal{A}$ and P is $< \varepsilon$.

In [5, sect. 4.1], lemma 6 states:
for any $\mathcal{A} \in \mathbb{M}$ containing a cycle, and $\varepsilon > 0$, in ε-neighborhood of $\mathcal{A}$, there exists an attractor $\mathcal{A}' \supseteq \mathcal{A}$, $\mathcal{A}' \in \mathbb{M}'$.

Thus, if we use the Hausdorf metric for $\mathbb{M}$, then we can state:
both $\mathbb{M}'$ and the aggregate $\mathbb{P}$ that consists of all cycles from $\mathbb{M}$ are dense in $\mathbb{M}$ (at least if we remove from $\mathbb{M}$ all attractors containing no cycles, about which we can not now say anything except that there exist very many too such attractors).

However, unfortunately, the proofs of the results represented in [4] were never published, although the proofs were completely represented in the author's doctoral thesis from 1966. The public defense of this thesis was held in May, 1967, exactly 50 years ago, and then, probably, it made sense to "stop", "look around" and, probably, join the more popular at this time topics, for example, smooth dynamical systems. Just at that time, and even in Kyiv, Smale appeared with his Smale horseshoe.

In 1971, the author was the main organizer of the summer school on dynamical systems. In 1972 and in 1976 (2d ed.), the lectures of this school was published and later even translated by the Amer. Math. Society (Alekseyev V.M., Katok A.B., Kushnirenko A.G., "Three papers on dynamical systems", AMS Transl.(2) **116**, 1981).

And already in 1973, author's PhD student V.S.Bondarchuk submitted his thesis "Invariant sets of smooth dynamical systems" ...

A substantial part of this thesis was represented in the papers (in Russian):

[6] Bondarchuk V.S., Sharkovsky A.N., The partially ordered system of omega-limit sets of expanding endomorphisms, in Dynamical systems and questions on the stability of the solutions of differential equations, Inst. Mat. Akad. Nauk USSR, Kiev, 1973, 128-164.

[7] Bondarchuk V.S., Sharkovsky A.N., Reconstructibility of expanding endomorphisms from the system of omega-limit sets, ibid., 28-34.

The papers [6] contained almost all statements from [4] and used the same way of proof as in sect. 4.1 of [5]. Thus, methods for proving the main results of [4] were actually published still in 1973 although already in application to a few other subjects. Another methods needed to be used by V.S.Bondarchuk were the method of Markov partions and topological Markov chains developed by Ya.G.Sinai and appropriately modernized by V.S.Bondarchuk. In [7], the algorithm for map reconstructing, developed in [5, sect. 4.2] was used; the map f has so many attractors (i.e., ω-limit sets), that the function $f(x)$ really can be restored (pointwise) if to have its attractors only !

σ-attractors

In dynamical systems theory, along with ω-limit sets of trajectories (attractors $\mathcal{A}$), are considered other sets that also characterize the asymptotic behavior of trajectories, for example, the so called "statistically limit set" of a trajectory, or σ-attractor $\mathcal{A}_\sigma(x)$, i.e., the smallest closed set such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \chi_U(f^i(x)) = 1$$

for any open neighborhood U of this set. For $x \in X$ always $\mathcal{A}_\sigma(x) \subseteq \mathcal{A}(x)$. In many cases, $\mathcal{A}_\sigma(x) \neq \mathcal{A}(x)$, so, if P is a cycle and $\Gamma(P)$ is a homoclinic trajectory to P, then $\mathcal{A}_\sigma(x) = P$ for every x such that $\mathcal{A}(x) = \Gamma(P)$.

Usually, it is more difficult to obtain descriptive estimates for the basins of $\mathcal{A}_\sigma$ than of $\mathcal{A}$. Even for the simplest intermixing one-dimensional map $x \mapsto 2x(mod\ 1)$, the question: "Is the set of points $x \in [0,1]$ for which $\mathcal{A}_\sigma(x) = \{0\}$, a set of the third Baire class?" remains open. For the same map, the set $\{x \in [0,1] : \mathcal{A}_x = [0,1]\}$, i.e. $\mathcal{B}([0,1])$, as well known, is a dense on $[0,1]$ G_δ set. What are σ-attractors for $x \in \mathcal{B}([0,1])$?

Read these!

[8] Sivak A.G., *Descriptive estimates for statistically limit sets of dynamical systems,* in *Dynamical Systems and Turbulence* [in Russian], Inst. Math., Kiev, 1989, 100-102.

[9] Sivak A.G., *On the structure of the set of trajectories generating an invariant measure,* in *Dynamical Systems and Nonlinear Phenomena* [in Russian], Inst. Math., Kiev, 1990, 39-43.

[10] Sivak A.G., *σ-Attractors of trajectories and their basins,* Addition in [5], sect.7, 281-310.

[11] Sharkovsky A.N., Sivak A.G., Basins of attractors of trajectories, J. Difference Eqns and Appl., **22**(2), 2016, 159-163.